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Section 8.1 *Basic Integration Rules*

Basic Integration Rules ($a > 0$)

1. $\int kf(u)\,du = k\int f(u)\,du$
2. $\int [f(u) \pm g(u)]\,du = \int f(u)\,du \pm \int g(u)\,du$
3. $\int du = u + C$
4. $\int u^n\,du = \dfrac{u^{n+1}}{n+1} + C, \quad n \neq -1$
5. $\int \dfrac{du}{u} = \ln|u| + C$
6. $\int e^u\,du = e^u + C$
7. $\int a^u\,du = \left(\dfrac{1}{\ln a}\right)a^u + C$
8. $\int \sin u\,du = -\cos u + C$
9. $\int \cos u\,du = \sin u + C$
10. $\int \tan u\,du = -\ln|\cos u| + C$
11. $\int \cot u\,du = \ln|\sin u| + C$
12. $\int \sec u\,du = \ln|\sec u + \tan u| + C$
13. $\int \csc u\,du = -\ln|\csc u + \cot u| + C$
14. $\int \sec^2 u\,du = \tan u + C$
15. $\int \csc^2 u\,du = -\cot u + C$
16. $\int \sec u \tan u\,du = \sec u + C$
17. $\int \csc u \cot u\,du = -\csc u + C$
18. $\int \dfrac{du}{\sqrt{a^2 - u^2}} = \arcsin\dfrac{u}{a} + C$
19. $\int \dfrac{du}{a^2 + u^2} = \dfrac{1}{a}\arctan\dfrac{u}{a} + C$
20. $\int \dfrac{du}{u\sqrt{u^2 - a^2}} = \dfrac{1}{a}\operatorname{arcsec}\dfrac{|u|}{a} + C$

Procedures for Fitting Integrands to Basic Rules

Technique	*Example*
Expand (numerator).	$(1 + e^x)^2 = 1 + 2e^x + e^{2x}$
Separate numerator.	$\dfrac{1+x}{x^2+1} = \dfrac{1}{x^2+1} + \dfrac{x}{x^2+1}$
Complete the square.	$\dfrac{1}{\sqrt{2x - x^2}} = \dfrac{1}{\sqrt{1 - (x-1)^2}}$
Divide improper rational function.	$\dfrac{x^2}{x^2+1} = 1 - \dfrac{1}{x^2+1}$
Add and subtract terms in numerator.	$\dfrac{2x}{x^2+2x+1} = \dfrac{2x+2-2}{x^2+2x+1} = \dfrac{2x+2}{x^2+2x+1} - \dfrac{2}{(x+1)^2}$
Use trigonometric identities.	$\cot^2 x = \csc^2 x - 1$
Multiply and divide by Pythagorean conjugate.	$\dfrac{1}{1+\sin x} = \left(\dfrac{1}{1+\sin x}\right)\left(\dfrac{1-\sin x}{1-\sin x}\right) = \dfrac{1-\sin x}{1-\sin^2 x} = \dfrac{1-\sin x}{\cos^2 x} = \sec^2 x - \dfrac{\sin x}{\cos^2 x}$

Ex.1 Integrate: $\int \frac{2x}{x-4}dx$

$= \int \left[2 + \frac{8}{x-4}\right] dx$

$= \int 2dx + 8\int \frac{1}{x-4}dx$

$= 2(x) + 8\int \frac{1}{u}\cdot du$

$= 2x + 8\cdot \ln|u| + C$

$\boxed{= 2x + 8\ln|x-4| + C}$

Long Division:

$$\begin{array}{r|l} & \quad 2 \quad R\,8 \\ x-4 & 2x+0 \\ & -(2x-8) \\ \hline & \quad 8 \end{array}$$

$\frac{2x}{x-4} = 2 + \frac{8}{x-4}$

Let $u = x-4$

$\frac{du}{dx} = 1$

$du = \frac{du}{dx}\cdot dx$

$du = 1\cdot dx$

$du = dx$

<u>check:</u> $\frac{d}{dx}\left[2x + 8\ln|x-4| + C\right]$

$= 2 + 8\cdot\left(\frac{1}{x-4}\right)\cdot\frac{d}{dx}(x-4) + 0$

$= 2 + \left(\frac{8}{x-4}\right)\cdot(1)$

$= 2 + \frac{8}{x-4} = \frac{2}{1}\left(\frac{x-4}{x-4}\right) + \frac{8}{x-4}$

$= \frac{2x-8+8}{x-4}$

$= \frac{2x}{x-4}$ ✓

Ex.2 Evaluate: $\int \sec(3x)\tan(3x)\,dx$

$= \int \sec(u)\tan(u)\cdot\left(\frac{du}{3}\right)$

$= \frac{1}{3}\cdot\int \sec(u)\tan(u)\,du$

$= \frac{1}{3}\cdot\left[\sec(u)\right] + C$

$\boxed{= \frac{1}{3}\sec(3x) + C}$

Let $u = 3x$

$\frac{du}{dx} = 3$

$du = \frac{du}{dx}\cdot dx$

$du = 3\cdot dx$

$\frac{du}{3} = dx$

<u>check:</u> $\frac{d}{dx}\left[\frac{1}{3}\sec(3x) + C\right]$

$= \frac{1}{3}\cdot\sec(3x)\tan(3x)\cdot\frac{d}{dx}(3x) + 0$

$= \frac{1}{3}\cdot\sec(3x)\tan(3x)\cdot 3 = \sec(3x)\tan(3x)$ ✓

Ex.3 Evaluate: $\int \csc^2(x)e^{\cot(x)}dx$

$= \int \csc^2(x) \cdot e^u \cdot \left[\frac{du}{-\csc^2(x)}\right]$

$= -\int e^u du$

$= -e^u + C$

$\boxed{= -e^{\cot(x)} + C}$

Let $u = \cot(x)$

$\frac{du}{dx} = -\csc^2(x)$

$du = \frac{du}{dx} \cdot dx$

$du = -\csc^2(x) \cdot dx$

$\frac{du}{-\csc^2(x)} = dx$

check: $\frac{d}{dx}\left[-e^{\cot(x)} + c\right]$

$= -e^{\cot(x)} \cdot \frac{d}{dx}[\cot(x)] + 0$

$= \left[-e^{\cot(x)}\right] \cdot \left[-\csc^2(x)\right]$

$= \csc^2(x)\, e^{\cot(x)}$ ✓

Ex.4 Evaluate: $\int \frac{5}{3e^x - 2}dx$

$= \int \left(\frac{5}{3u-2}\right)\left(\frac{du}{u}\right)$

$= 5\int \frac{1}{u(3u-2)} du$

$= \frac{5}{2}\int \frac{2}{u(3u-2)} du$

$= \frac{5}{2}\int \frac{3u - 3u + 2}{u(3u-2)} du$

$= \frac{5}{2}\int \frac{3u - (3u-2)}{u \cdot (3u-2)} du$

$= \frac{5}{2}\int \frac{3u}{u(3u-2)} du - \frac{5}{2}\int \frac{(3u-2)}{u(3u-2)} du = \frac{5}{2}\int \frac{3}{3u-2} du - \frac{5}{2}\int \frac{1}{u} du$

$= \frac{5}{2}\int \frac{3}{z}\left(\frac{dz}{3}\right) - \frac{5}{2} \cdot \ln|u| + c$

$= \frac{5}{2}\int \frac{1}{z} dz - \frac{5}{2}\ln|e^x| + c$

$= \frac{5}{2} \cdot \ln|z| - \frac{5}{2}\ln|e^x| + c$

$= \frac{5}{2}\ln|3u-2| - \frac{5}{2}\ln|e^x| + c$

$= \frac{5}{2}\ln|3e^x - 2| - \frac{5}{2}\ln|e^x| + c$

$= \frac{5}{2}\left[\ln|3e^x - 2| - \ln|e^x|\right] + c$

$= \frac{5}{2}\left[\ln\left|\frac{3e^x - 2}{e^x}\right|\right] + c$

$\boxed{= \frac{5}{2}\ln\left|\frac{3e^x-2}{e^x}\right| + c}$ ✓

Let $z = 3u - 2$

$\frac{dz}{du} = 3$

$dz = \frac{dz}{du} \cdot du$

$dz = 3 \cdot du$

$\frac{dz}{3} = du$

Let $u = e^x$

$\frac{du}{dx} = e^x$

$du = \frac{du}{dx} \cdot dx$

$du = e^x dx$

$du = u\, dx$

$\frac{du}{u} = dx$

OR

$\ln(u) = \ln(e^x)$

$\ln(u) = x$

$\frac{d}{dx}[\ln(u)] = \frac{d}{dx}(x)$

$\frac{1}{u} \cdot \frac{du}{dx} = 1$

$\frac{du}{dx} = u$

$du = \frac{du}{dx} \cdot dx$

$du = u \cdot dx$

$\frac{du}{u} = dx$

Ex.5 Integrate: $\int \frac{1}{\cos(\theta)-1} d\theta$

$= \int \frac{1}{-1\cdot[1-\cos(\theta)]} d\theta$

$= -\int \frac{1}{1-\cos(\theta)} d\theta$

$= -\int \frac{1}{2\sin^2(\frac{\theta}{2})} d\theta$

$= -\int \frac{1}{2\sin^2(u)} \cdot (2du)$

$= -\int \csc^2(u)\, du$

$= -[-\cot(u)] + c$

$\boxed{= \cot(\frac{\theta}{2}) + C}$

"Power Reducing Identity"

Trig. Identity

$\sin^2(\alpha) = \frac{1-\cos(2\alpha)}{2}$

$2\sin^2(\alpha) = 1-\cos(2\alpha)$

$2\alpha = \theta$, $\alpha = \frac{\theta}{2}$

$1-\cos(\theta) = 2\sin^2(\frac{\theta}{2})$

Let $u = \frac{\theta}{2}$

$\frac{du}{d\theta} = \frac{1}{2}$

$du = \frac{du}{d\theta} \cdot d\theta$

$du = \frac{1}{2} d\theta$

$2du = d\theta$

OR

"conjugates"

$\cos^2(\theta) + \sin^2(\theta) = 1$

$\cos^2(\theta) - 1 = -\sin^2(\theta)$

$= \int \frac{1}{\cos(\theta)-1} d\theta = \int \frac{1}{[\cos(\theta)-1]} \cdot \frac{[\cos(\theta)+1]}{[\cos(\theta)+1]} d\theta = \int \frac{\cos(\theta)+1}{\cos^2\theta - 1} d\theta$

$= \int \frac{\cos(\theta)+1}{-\sin^2(\theta)} d\theta = -\int \frac{\cos(\theta)}{\sin^2(\theta)} d\theta - \int \frac{1}{\sin^2(\theta)} d\theta$

$= -\int \frac{\cos(\theta)}{u^2} \cdot \left(\frac{du}{\cos(\theta)}\right) - \int \csc^2(\theta) d\theta = -\int u^{-2} du - [-\cot(\theta)] + c$

$= -[-u^{-1}] + \cot(\theta) + c = \frac{1}{u} + \frac{\cos(\theta)}{\sin(\theta)} + c = \frac{1}{\sin(\theta)} + \frac{\cos(\theta)}{\sin(\theta)} + C$

$\boxed{= \frac{1+\cos(\theta)}{\sin(\theta)} + c}$

Let $u = \sin(\theta)$

$\frac{du}{d\theta} = \cos(\theta)$

$du = \frac{du}{d\theta} \cdot d\theta$

$du = \cos(\theta) d\theta$

$\frac{du}{\cos(\theta)} = d\theta$

We have $\boxed{\cos(\frac{\theta}{2}) + c = \frac{1+\cos(\theta)}{\sin(\theta)} + C}$!

Ex.6 Evaluate: $\displaystyle\int \frac{2}{3(\sec(x)-1)}dx$

"conjugates"

$$= \frac{2}{3}\cdot\int \frac{1}{[\sec(x)-1]}\cdot\left[\frac{\sec(x)+1}{\sec(x)+1}\right]dx$$

$$= \frac{2}{3}\int \frac{\sec(x)+1}{\sec^2(x)-1}dx$$

$$= \frac{2}{3}\int \frac{\sec(x)+1}{\tan^2(x)}dx$$

$$= \frac{2}{3}\int \frac{\sec(x)}{\tan^2(x)}dx + \frac{2}{3}\int \frac{1}{\tan^2(x)}dx$$

$$= \frac{2}{3}\int \left[\frac{1}{\cos(x)}\right]\cdot\left[\frac{\cos^2(x)}{\sin^2(x)}\right]dx + \frac{2}{3}\int \cot^2(x)dx$$

$$= \frac{2}{3}\int \frac{\cos(x)}{\sin^2(x)}dx + \frac{2}{3}\int [\csc^2(x)-1]dx$$

$$= \frac{2}{3}\int \frac{\cos(x)}{(u)^2}\cdot\left[\frac{du}{\cos(x)}\right] + \frac{2}{3}\int \csc^2(x)\,dx - \frac{2}{3}\int 1\,dx$$

$$= \frac{2}{3}\int u^{-2}du + \frac{2}{3}[-\cot(x)] - \frac{2}{3}(x) + C$$

$$= \frac{2}{3}[-u^{-1}] - \frac{2}{3}\cot(x) - \frac{2x}{3} + C$$

$$= -\frac{2}{3}\left(\frac{1}{u}\right) - \frac{2}{3}\cot(x) - \frac{2}{3}x + C$$

$$= -\frac{2}{3}\left[\frac{1}{\sin(x)}\right] - \frac{2}{3}\cot(x) - \frac{2}{3}x + C$$

$$\boxed{= -\frac{2}{3}[\csc(x) + \cot(x) + x] + C}$$

Pythagorean Identities!

$\sec^2(x) - 1 = \tan^2(x)$

$\cot^2(x) = \csc^2(x) - 1$

Let $u = \sin(x)$

$$\frac{du}{dx} = \cos(x)$$

$$du = \frac{du}{dx}\cdot dx$$

$$du = \cos(x)\,dx$$

$$\frac{du}{\cos(x)} = dx$$

Ex.7 Integrate: $\displaystyle\int \frac{1}{(x-1)\sqrt{4x^2-8x+3}}\,dx$

$$= \int \frac{1}{(x-1)\sqrt{[2(x-1)]^2-1}}\,dx$$

"Difference of squares"

$$= \int \frac{1}{\left(\frac{u}{2}\right)\sqrt{(u)^2-1}} \cdot \left(\frac{du}{2}\right)$$

$$= \int \frac{1}{u\sqrt{u^2-1}}\,du$$

$$= \frac{1}{a}\operatorname{arcsec}\left(\frac{|u|}{a}\right) + C \quad , a=1$$

$$= \frac{1}{1}\operatorname{arcsec}\left(\frac{|2(x-1)|}{1}\right) + C$$

$$\boxed{= \operatorname{arcsec}|2(x-1)| + C}$$

"complete the square"

$4x^2-8x+3$
$= 4(x^2-2x)+3$
$= 4(x^2-2x+1)+(-4)+3$
$\left[\frac{1}{2}(-2)\right]^2 = (-1)^2 = 1$
$= 4(x-1)^2-1$
$= [2(x-1)]^2-1$

Let $u = 2(x-1)$
$\frac{du}{dx} = 2$
$du = \frac{du}{dx}\,dx$
$du = 2\,dx$
$\frac{du}{2} = dx$

$u = 2(x-1)$
$\frac{u}{2} = x-1$

Ex.8 Evaluate: $\displaystyle\int_1^e \frac{1-\ln(x)}{x}\,dx$

$$= \int_{u=1}^{u=0} \frac{(u)}{x} \cdot (-x\,du) = -\int_1^0 u\,du$$

$$= \int_0^1 u\,du = \left[\frac{u^2}{2}\right]_0^1$$

$$= \frac{1}{2} - 0 = \boxed{\frac{1}{2}}$$

Let $u = 1-\ln(x)$
$\frac{du}{dx} = -\frac{1}{x}$
$du = \frac{du}{dx}\cdot dx$
$du = -\frac{1}{x}\,dx$
$-x\,du = dx$

$x=e$, $u = 1-\ln(e)$
$u = 1-1$
$u = 0$

$x=1$, $u = 1-\ln(1)$
$u = 1-0$
$u = 1$